Cutting-Planes in Mixed-Integer Linear Programs: An Overview

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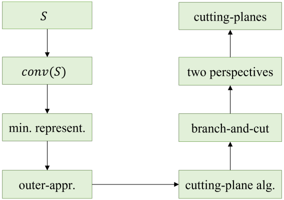
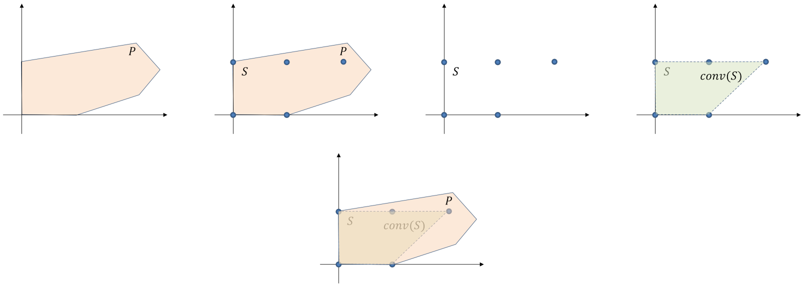
Consider a mixed-integer linear program:

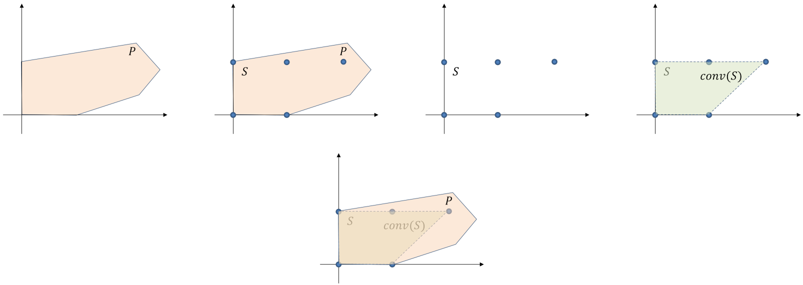
$$\begin{aligned} \min_{x} \quad & c^\top x \\ \text{s.t.} \quad & x \in S, \end{aligned} \tag{1}$$

where $S \equiv \{x \in P : x_i \in \mathbb{Z}\ \forall i \in I\}, P \equiv \{x \in \mathbb{R}^n_+ : Ax \leq b\}, I \subseteq [n]$.
Let $conv(S)$ denote the convex hull of S.





$$S \subseteq conv(S) \subseteq P$$



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Observation

$conv(S)$ is a polyhedron, i.e., $conv(S)$ can be represented via finitely many linear inequalities. (Prove it!)

Problem (1) is equivalent to the following linear program

$$\begin{aligned} \min_{x} \quad & c^\top x \\ \text{s.t.} \quad & x \in conv(S) \end{aligned}$$

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Characterizing $conv(S)$

Since $conv(S)$ is a polyhedron, we can characterize $conv(S)$ from:

- an algebraic perspective: H-polyhedron/H-representation

$$conv(S) = \left\{ x \in \mathbb{R}^n : \tilde{A}x \leq \tilde{b} \right\}$$

- a geometric perspective: V-polyhedron/V-representation

$$conv(S) = \underbrace{\sum_j \alpha_j v^j}_{\text{extreme points}} + \underbrace{\sum_\ell \beta_\ell r^\ell}_{\text{extreme rays}}, \qquad \text{Minkowski-Weyl Theorem}$$

where $\sum_j \alpha_j = 1$ and $\alpha_j, \beta_\ell \geq 0$.

For generality purposes, we will proceed with an algebraic perspective and aim for a minimal representation of $conv(S)$.

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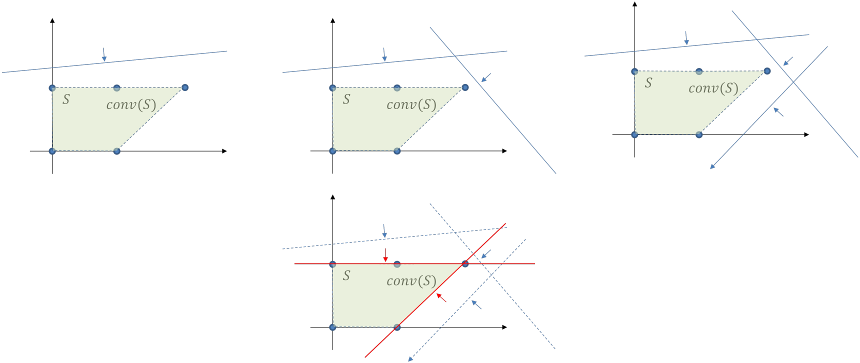
Minimal representation



The minimal representation of $conv(S)$ entails facet-defining inequalities.

Facet-defining inequalities

$\alpha^\top x \leq \beta$ is a valid inequality for set X if $\alpha^\top \bar{x} \leq \beta \quad \forall \bar{x} \in X$.

Definition (Face)

A face of a polyhedron $X \subseteq \mathbb{R}^n$ is a set of the form:

$$F \equiv X \cap \left\{ x \in \mathbb{R}^n : \alpha^\top x = \beta \right\},$$

where $\alpha^\top x \leq \beta$ is a valid inequality for X.

Definition (Facet)

Face F of set X is a facet iff $dim(F) = dim(X) - 1$.

Any valid inequality for set X that defines a facet is called a facet-defining inequality.

Question

Since problem (4) is an LP and can be solved in polynomial time, does it mean that problem (1) can be solved in polynomial time? In another word, does an general MILP have a complexity of $\mathcal{P}$?

Answer

Of course not, characterizing $conv(S)$ is $\mathcal{NP}$-hard.

- The size of $conv(S)$ is huge, i.e., the number of facets for $conv(S)$ is of an exponential size.
- Identifying a violated facet-defining inequality of $conv(S)$ is as hard as solving the original problem (1).

We try to outer-approximate $conv(S)$, rather than search for a minimal representation of $conv(S)$.

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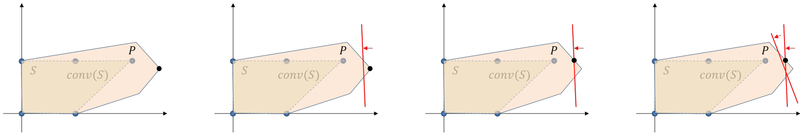
Consider an LP relaxation of problem (1):

$$\begin{aligned} \min_{x} \quad & c^\top x \\ \text{s.t.} \quad & x \in P. \end{aligned} \tag{2}$$

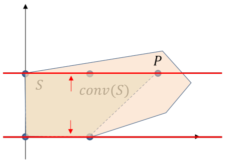
The cutting-plane algorithm works as follows:

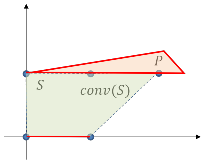
i. Obtain the optimal solution $\bar{x}$ to problem (2). If $\bar{x} \in S$, then stop. Otherwise, go to step ii.

ii. Identify an inequality $\alpha^\top x \leq \beta$ valid for S (or equivalently $conv(S)$) such that $\alpha^\top \bar{x} > \beta$. Take $P \leftarrow P \cap \{x \in \mathbb{R}^n : \alpha^\top x \leq \beta\}$, go to step i.



- Guarantee to identify a violated cut?
  - Yes.
- The complexity of separation problem?
  - manageable
- Will this algorithm terminate?
  - Yes, but with a slow convergence rate.
- Numerical error
  - Branch-and-Cut, readers are referred to for an amazing story [C+07].





Branch-and-Cut

- [AW13] identifies 4 core components based on: presolve, primal heuristics, cutting-planes, and branching
- [ABG+20] concludes that presolve together with cutting-plane techniques are by far the most important individual tools contributing to the power of modern MIP solvers.

Cutting-planes

We attempt to answer the following questions:

- How to generate cuts?
- Cutting-planes in the literature/MIP solvers
- ~~Strength of cutting-planes~~
- ~~How to strengthen/lift cuts if possible?~~
- ~~Which cuts are selected?~~ [Ach09]

References: [Cor08], IP course notes by Ted Ralphs

Generating cutting-planes

- algebraic perspective: e.g., Chvatal procedure
  i. Take combinations of the known valid inequalities.
  ii. Use rounding to produce stronger ones
- geometric perspective: e.g., split inequalities
  i. Use a disjunction to generate several disjoint polyhedra whose union contains S
  ii. Generate inequalities valid for the convex hull of this union

Given $S \equiv \{x \in \mathbb{Z}^{|I|} \times \mathbb{R}^{|C|} : Ax \leq b\}$ and any $u \in \mathbb{R}^m_+$, we have

$$u^\top Ax \leq u^\top b$$
$$u^\top A_I x_I + u^\top A_C x_C \leq u^\top b$$

If $u^\top A_C = 0, u^\top A_I \in \mathbb{Z}^{|I|}$ and $u^\top b \notin \mathbb{Z}$, then

$$u^\top A_I x_I \leq u^\top b$$
$$\sum_{i \in I} u^\top A_i x_i \leq \lfloor u^\top b \rfloor$$

The resulting inequality is called a "Chvatal inequality".

Algebraic: Chvatal procedure

Given $S \equiv \{x \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+^{|C|} : Ax \leq b\}$ and any $u \in \mathbb{R}_+^m$, we have

$$u^\top Ax \leq u^\top b$$
$$u^\top A_I x_I + u^\top A_C x_C \leq u^\top b$$

If $u^\top A_C \geq 0$ and $u^\top b \notin \mathbb{Z}$, then

$$u^\top A_I x_I \leq u^\top b$$
$$\sum_{i \in I} \lfloor u^\top A_i \rfloor x_i \leq u^\top b$$
$$\sum_{i \in I} \lfloor u^\top A_i \rfloor x_i \leq \lfloor u^\top b \rfloor$$

- The resulting inequality is called a "Chvatal-Gomory inequality".
- Clearly, the "Chvatal inequality" is a special case of "Chvatal-Gomory inequalities".

Geometric: split inequalities

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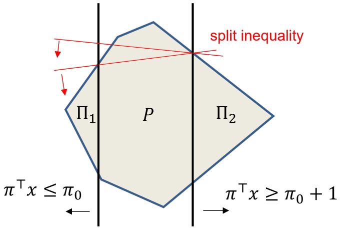
Definition (Split inequalities)

Given P and S, an inequality $\alpha^\top x \leq \beta$ is a split inequality if there exists a split (π, π_0) with $\pi_I \in \mathbb{Z}^{|I|}, \pi_C = 0$ and $\pi_0 \in \mathbb{Z}$ such that $\alpha^\top x \leq \beta$ is valid for both sets

$$\Pi_1 \equiv \left\{ x \in P : \pi^\top x \leq \pi_0 \right\},$$
$$\Pi_2 \equiv \left\{ x \in P : \pi^\top x \geq \pi_0 + 1 \right\}.$$



Geometric: split inequalities

Examples of split inequalities:

- Gomory fractional/mixed-integer
- Mixed-integer rounding
- Chvatal-Gomory (a special case when $\Pi_2 = \emptyset$)
- lift-and-project (disjunctive cuts might not be)

General-purpose cuts

1. Gomory
   - ~~Gomory's fractional~~ (for pure integer programming) [Gom10] (CPLEX)
   - Gomory's mixed integer [BCCN96]
2. mixed integer rounding [MW01]
3. Chvatal-Gomory
   - strong CG [LL02]
   - zero-half [CF96]
   - mod-k [CFL00]
4. disjunctive programming
   - disjunctive [Bal18] (CPLEX)
   - lift-and-project [BB09] (for mixed 0/1 programming)

5. infeasibility proof (conflict analysis) [WBH21]
6. RLT [SA98]: reformulation-linearization techniques
7. sub-MIP: no good cuts
8. relax-and-lift [BM14]: i. construct a relaxation of an LP row by fixing variables to its bounds; ii. try to lift binaries into the cut. (Gurobi)
9. learned: found incidentally while executing some presolve procedure, not specific to any model structure. (Gurobi)

Structural cuts

1. cover
   - cover [HGH+19]
   - flow cover [PVRW85, VRW86, GNS99]
   - GUB cover [Wol90]
2. implied bound
   - implied bound [Sav94]
   - projected implied bound
3. clique [JP82, Sav94]
4. network
   - flow path [VRW85, OW03, Chr09]
   - multi-commodity flow [AR10]

[ABG+20] Tobias Achterberg, Robert E Bixby, Zonghao Gu, Edward Rothberg, and Dieter Weninger, *Presolve reductions in mixed integer programming*, INFORMS Journal on Computing **32** (2020), no. 2, 473–506.

[Ach09] Tobias Achterberg, *Scip: solving constraint integer programs*, Mathematical Programming Computation **1** (2009), no. 1, 1–41.

[AR10] Tobias Achterberg and Christian Raack, *The mcf-separator: detecting and exploiting multi-commodity flow structures in mips*, Mathematical Programming Computation **2** (2010), no. 2, 125–165.

[AW13] Tobias Achterberg and Roland Wunderling, *Mixed integer programming: Analyzing 12 years of progress*, Facets of combinatorial optimization, Springer, 2013, pp. 449–481.

[Bal18] Egon Balas, *Disjunctive programming*, Springer, 2018.

[BB09] Egon Balas and Pierre Bonami, *Generating lift-and-project cuts from the lp simplex tableau: open source implementation and testing of new variants*, Mathematical Programming Computation **1** (2009), no. 2-3, 165–199.

[BCCN96] Egon Balas, Sebastian Ceria, Gérard Cornuéjols, and N Natraj, *Gomory cuts revisited*, Operations Research Letters **19** (1996), no. 1, 1–9.

[BM14] Pierre Bonami and Michel Minoux, *A comparison of some valid inequality generation methods for general 0–1 problems*, Applications of Combinatorial Optimization (2014), 49–72.

[C+07] Gérard Cornuéjols et al., *Revival of the gomory cuts in the 1990's.*, Annals of Operations Research **149** (2007), no. 1, 63–66.

[CF96] Alberto Caprara and Matteo Fischetti, {*0, 1/2*}*-chvátal-gomory cuts*, Mathematical Programming **74** (1996), no. 3, 221–235.

[CFL00] Alberto Caprara, Matteo Fischetti, and Adam N Letchford, *On the separation of maximally violated mod-k cuts*, Mathematical Programming **87** (2000), no. 1, 37–56.

[Chr09] Philipp M Christophel, *Separation algorithms for cutting planes based on mixed integer row relaxations*, Ph.D. thesis, Ph. D. Thesis, Universität Paderborn, Paderborn, 2009.

[Cor08] Gérard Cornuéjols, *Valid inequalities for mixed integer linear programs*, Mathematical programming **112** (2008), no. 1, 3–44.

[GNS99] Zonghao Gu, George L Nemhauser, and Martin WP Savelsbergh, *Lifted flow cover inequalities for mixed 0-1 integer programs*, Mathematical Programming **85** (1999), no. 3, 439–467.

References V

[Gom10] Ralph E Gomory, *Outline of an algorithm for integer solutions to linear programs and an algorithm for the mixed integer problem*, 50 Years of Integer Programming 1958-2008, Springer, 2010, pp. 77–103.

[HGH+19] Christopher Hojny, Tristan Gally, Oliver Habeck, Hendrik Lüthen, Frederic Matter, Marc E Pfetsch, and Andreas Schmitt, *Knapsack polytopes: a survey*, Annals of Operations Research (2019), 1–49.

[JP82] Ellis L Johnson and Manfred W Padberg, *Degree-two inequalities, clique facets, and biperfect graphs*, North-Holland Mathematics Studies, vol. 66, Elsevier, 1982, pp. 169–187.

[LL02] Adam N Letchford and Andrea Lodi, *Strengthening chvátal–gomory cuts and gomory fractional cuts*, Operations Research Letters **30** (2002), no. 2, 74–82.

[MW01] Hugues Marchand and Laurence A Wolsey, *Aggregation and mixed integer rounding to solve mips*, Operations research **49** (2001), no. 3, 363–371.

[OW03] Francisco Ortega and Laurence A Wolsey, *A branch-and-cut algorithm for the single-commodity, uncapacitated, fixed-charge network flow problem*, Networks: An International Journal **41** (2003), no. 3, 143–158.

[PVRW85] Manfred W Padberg, Tony J Van Roy, and Laurence A Wolsey, *Valid linear inequalities for fixed charge problems*, Operations Research **33** (1985), no. 4, 842–861.

References VII

[SA98] Hanif D Sherali and Warren P Adams, *Reformulation-linearization techniques for discrete optimization problems*, Handbook of Combinatorial Optimization: Volume1–3 (1998), 479–532.

[Sav94] Martin WP Savelsbergh, *Preprocessing and probing techniques for mixed integer programming problems*, ORSA Journal on Computing **6** (1994), no. 4, 445–454.

[VRW85] Tony J Van Roy and Laurence A Wolsey, *Valid inequalities and separation for uncapacitated fixed charge networks*, Operations Research Letters **4** (1985), no. 3, 105–112.

[VRW86] ______, *Valid inequalities for mixed 0–1 programs*, Discrete Applied Mathematics **14** (1986), no. 2, 199–213.

References VIII

[WBH21] Jakob Witzig, Timo Berthold, and Stefan Heinz, *Computational aspects of infeasibility analysis in mixed integer programming*, Mathematical Programming Computation **13** (2021), no. 4, 753–785.

[Wol90] Laurence A Wolsey, *Valid inequalities for 0–1 knapsacks and mips with generalised upper bound constraints*, Discrete Applied Mathematics **29** (1990), no. 2-3, 251–261.