An Introduction to Gomory Cuts

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Consider a mixed-integer linear program:

$$\begin{aligned} \min_{x} \quad & c^\top x \\ \text{s.t.} \quad & x \in S, \end{aligned} \tag{1}$$

where $S \equiv \{x \in P : x_i \in \mathbb{Z}\ \forall i \in I\}, P \equiv \left\{x \in \mathbb{R}^n_+ : Ax = b\right\}, I \subseteq [n]$.

We aim to identify Gomory cuts for problem (1).

Who is Ralph E. Gomory?

Check out http://www.ralphgomory.org/.

Outline

- Gomory fractional inequalities (for pure integer programming) [Gom10]
- Gomory mixed-integer inequalities (GMI)
- The relationship between GMI and split inequalities (~~MIR and lift-and-project~~)
- How to strengthen GMI

Reading materials: [C+07, Cor08, Fuk10, CCZ+14]

Generating cutting-planes

In our previous talk, we discussed two perspectives for generating valid inequalities for MILPs:

- algebraic perspective: e.g., Chvatal procedure
  i. Take combinations of the known valid inequalities.
  ii. Use rounding to produce stronger ones
- geometric perspective: e.g., split inequalities
  i. Use a disjunction to generate several disjoint polyhedra whose union contains S
  ii. Generate inequalities valid for the convex hull of this union

Remark

The geometric perspective provides a natural way to strengthen those generated inequalities.

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Gomory fractional inequalities

Given a pure integer program (i.e., $I = [n]$), consider a simplex tableau

$$x_i + \sum_{j \in N} \bar{a}_{ij} x_j = \bar{a}_{i0} \quad \forall i \in B. \tag{2}$$

If $\bar{a}_{i0} \notin \mathbb{Z}$ for some $i \in B$, then apply the Chavatal-Gomory procedure

$$x_i + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq \lfloor \bar{a}_{i0} \rfloor. \tag{3}$$

By subtracting (3) from (2), we obtain

$$\sum_{j \in N} f_{ij} x_j \geq f_{i0}, \tag{4}$$

where $f_{ij} = \bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor$. (4) is called a Gomory fractional inequality.

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Gomory fractional inequalities

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Example

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\begin{aligned}
\max_{x} \quad & 5.5x_1 + 2.1x_2 \\
\text{s.t.} \quad & -x_1 + x_2 \leq 2 \\
& 8x_1 + 2x_2 \leq 17 \\
& x_1, x_2 \in \mathbb{Z}_+
\end{aligned} \tag{5}
$$

$$x_2 + 0.8x_3 + 0.1x_4 = 3.3 \Longrightarrow \frac{8}{3}x_3 + \frac{1}{3}x_4 \geq 1$$

Remarks

- Gomory fractional inequality (4) always cuts off the current solution $\bar{x}$.
- [Gom10, WN99] proposed a finite cutting plane algorithm (i.e., lexicographic dual simplex) for pure integer programming problems using Gomory fractional cuts.

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Given a general MILP, consider a simplex tableau

$$x_i = \bar{a}_{i0} - \sum_{j \in N} \bar{a}_{ij} x_j \quad \forall i \in B. \tag{6}$$

We can rewrite the RHS of (6) as

$$\boxed{\lfloor \bar{a}_{i0} \rfloor + f_{i0} - \sum_{\substack{j \in N \cap I: \\ f_{ij} \le f_{i0}}} (\lfloor \bar{a}_{ij} \rfloor + f_{ij}) x_j - \sum_{\substack{j \in N \cap I: \\ f_{ij} > f_{i0}}} (\lceil \bar{a}_{ij} \rceil - 1 + f_{ij}) x_j - \sum_{j \in N \cap C} \bar{a}_{ij} x_j}$$

As a result,

$$\boxed{z \equiv f_{i0} - \sum_{\substack{j \in N \cap I: \\ f_{ij} \le f_{i0}}} f_{ij} x_j - \sum_{\substack{j \in N \cap I: \\ f_{ij} > f_{i0}}} (f_{ij} - 1) x_j - \sum_{j \in N \cap C} \bar{a}_{ij} x_j \text{ is an integer}}$$

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Hence, the following disjunction must be valid.

$$z \leq 0 \vee z \geq 1 \tag{7}$$

Simplify these inequalities, we have

$$\begin{aligned}
&\sum_{\substack{j \in N \cap I: \\ f_{ij} \leq f_{i0}}} \frac{-f_{ij}}{1-f_{i0}} x_j + \sum_{\substack{j \in N \cap I: \\ f_{ij} > f_{i0}}} \frac{1-f_{ij}}{1-f_{i0}} x_j + \sum_{j \in N \cap C} \frac{-\bar{a}_{ij}}{1-f_{i0}} x_j \geq 1 \\
&\sum_{\substack{j \in N \cap I: \\ f_{ij} \leq f_{i0}}} \frac{f_{ij}}{f_{i0}} x_j + \sum_{\substack{j \in N \cap I: \\ f_{ij} > f_{i0}}} \frac{f_{ij}-1}{f_{i0}} x_j + \sum_{j \in N \cap C} \frac{\bar{a}_{ij}}{f_{i0}} x_j \geq 1
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One can have the following by combining inequalities (8)

Gomory's Mixed-Integer Cuts

$$\sum_{\substack{j\in N\cap I:\\ f_{ij}\le f_{i0}}} \frac{f_{ij}}{f_{i0}} x_j + \sum_{\substack{j\in N\cap I:\\ f_{ij}> f_{i0}}} \frac{1-f_{ij}}{1-f_{i0}} x_j + \sum_{\substack{j\in N\cap C:\\ \bar{a}_{ij}<0}} \frac{-\bar{a}_{ij}}{1-f_{i0}} x_j + \sum_{\substack{j\in N\cap C:\\ \bar{a}_{ij}\ge 0}} \frac{\bar{a}_{ij}}{f_{i0}} x_j \ge 1 \tag{9}$$

Example

Consider problem (5), we have a row of its simplex tableau

$$x_2 + 0.8x_3 + 0.1x_4 = 3.3 \Longrightarrow \frac{2}{7}x_3 + \frac{1}{3}x_4 \ge 1$$

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$$x_2 + 0.8x_3 + 0.1x_4 = 3.3 \Longrightarrow \frac{2}{7}x_3 + \frac{1}{3}x_4 \ge 1$$

Remarks

- The inequality (10) always cuts off the basic feasible solution $\bar{x}$.
- One can rewrite inequality (9) as

$$\sum_{j \in N \cap I} \min \left\{ \frac{f_{ij}}{f_{i0}}, \frac{1 - f_{ij}}{1 - f_{i0}} \right\} x_j + \sum_{j \in N \cap C} \max \left\{ \frac{-\bar{a}_{ij}}{1 - f_{i0}}, \frac{\bar{a}_{ij}}{f_{i0}} \right\} x_j \geq 1 \tag{10}$$

- For pure integer programs, GMI (9) dominates Gomory fractional inequality (4).
- When rewriting (6), one can replace $\bar{a}_{ij}$ by $\lfloor \bar{a}_{ij} \rfloor + f_{ij}$ for each $j \in N \cap I$. However, the resulting cuts will be dominated by (9).
- When formulating (7), one may want to use $z \leq k \ \vee \ z \geq k + 1$ where $k \neq 0$. This would fail to produce a desirable system.
- A finite cutting-plane algorithm for solving MILPs via GMI does not exist [WN99].

Basis and corner polyhedron

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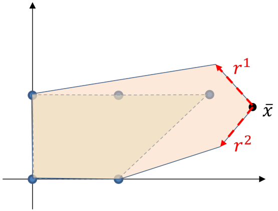
Consider a simplex tableau that corresponds to basis B:

$$x_i = \bar{a}_{i0} + \sum_{j \in N} -\bar{a}_{ij} x_j \quad \forall i \in B.$$

The corner polyhedron associated with B is given by

$$\begin{aligned} P(B) &\equiv \bar{x} + Cone(\{r^j\}_{j \in N}) \\ &= \{x \in \mathbb{R}^n : Ax = b, x_j \geq 0 \ \forall j \in N\}. \end{aligned}$$



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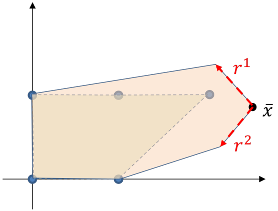
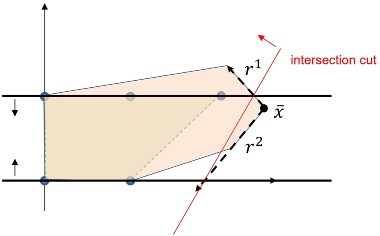


Consider a convex set K such that $\bar{x} \in int(K)$ and $int(K) \cap S = \emptyset$, then one can generate an intersection cut given by

$$\sum_{j \in N} \frac{x_j}{\alpha_j(K)} \geq 1,$$

where $\alpha_j(K) \equiv \max_{\alpha} \{\alpha : \bar{x} + \alpha r^j \in K\}$.



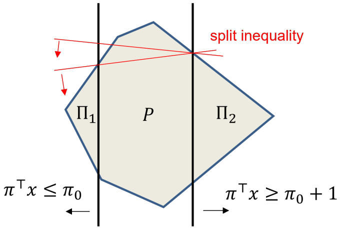
Interested readers are referred to "An Introduction to Intersection Cut and Their Applications".

Split inequalities

Definition (Split inequalities)

Given P and S, an inequality $\alpha^\top x \le \beta$ is a split inequality if there exists a split (π, π_0) with $\pi_I \in \mathbb{Z}^p, \pi_C = 0$ and $\pi_0 \in \mathbb{Z}$ such that $\alpha^\top x \le \beta$ is valid for both sets

$$\Pi_1 \equiv \left\{x \in P : \pi^\top x \le \pi_0\right\},$$
$$\Pi_2 \equiv \left\{x \in P : \pi^\top x \ge \pi_0 + 1\right\}.$$



Generating intersec. cuts from a split

Consider an intersection cut induced by a convex set

$$\left\{x \in \mathbb{R}^n : \pi_0 \le \pi^\top x \le \pi_0 + 1\right\},$$

where (π, π_0) is a split with $\pi_0 \equiv \lfloor \pi^\top \bar{x} \rfloor$, we can compute

$$\alpha_j = \max_{\alpha} \left\{\alpha : \pi_0 \le \pi^\top (\bar{x} + \alpha r^j) \le \pi_0 + 1\right\}$$
$$= \begin{cases} \frac{1-\epsilon(\pi,\pi_0)}{\pi^\top r^j} & \text{if } \pi^\top r^j > 0, \\ \frac{\epsilon(\pi,\pi_0)}{-\pi^\top r^j} & \text{if } \pi^\top r^j < 0, \\ +\infty & \text{otherwise.} \end{cases}$$

where $\epsilon(\pi, \pi_0) \equiv \pi^\top \bar{x} - \pi_0$.

GMI as split ineq.

Theorem

The GMI cut obtained from the row of the simplex tableau, in which x_i is basic, is the same as the intersection cut induced by a convex set $\{x \in \mathbb{R}^n : \pi_0 \leq \pi^\top x \leq \pi_0 + 1\}$, where

$$\pi_j \equiv \begin{cases} \lfloor \bar{a}_{ij} \rfloor & \text{if } j \in N \text{ and } f_{ij} \leq f_{i0}, \\ \lceil \bar{a}_{ij} \rceil & \text{if } j \in N \text{ and } f_{ij} > f_{i0}, \\ 1 & \text{if } j = i, \\ 0 & \text{otherwise,} \end{cases}$$

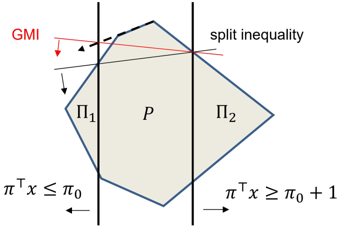
$$\pi_0 \equiv \lfloor \pi^\top \bar{x} \rfloor.$$

Proof.

$$\epsilon(\pi, \pi_0) = \pi^\top \bar{x} - \pi_0 = \bar{x}_i - \lfloor \bar{x}_i \rfloor = f_{i0}$$

$$\pi^\top r^j = \pi_B^\top r_B^j + \pi_N^\top r_N^j = \begin{cases} \lfloor \bar{a}_{ij} \rfloor - \bar{a}_{ij} = -f_{ij} & \text{if } j \in N \cap I : f_{ij} \le f_{i0} \\ \lceil \bar{a}_{ij} \rceil - \bar{a}_{ij} = 1 - f_{ij} & \text{if } j \in N \cap I : f_{ij} > f_{i0} \\ -\bar{a}_{ij} & \text{if } j \in N \cap C \end{cases}$$

□



Strengthen GMI

There are two perspectives to strengthen GMI:

- modify the basis while keeping the disjunction fixed
  - lift-and-project cut [BP03]
- modify the disjunction while keeping the basis fixed
  - reduce-and-split cut [ACL05]

One may combine the above two perspectives to further strengthen GMI:

- pivot-and-reduce [WKS11]
- alternate between lift-and-project and reduce-and-split [BCKN13]

In this talk, we will only introduce reduce-and-split cuts.

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Motivation: for $j \in N \cap C$, smaller $|\bar{a}_{ij}|$ values might lead to stronger GMI cuts.

Consider two rows of a simplex tableau:

$$x_i = \bar{a}_{i0} - \sum_{j \in N} \bar{a}_{ij} x_j, \; x_k = \bar{a}_{k0} - \sum_{j \in N} \bar{a}_{kj} x_j.$$

Given any $\delta \in \mathbb{Z}$, we obtain

$$x_i + \delta x_k = \bar{a}_{i0} + \delta \bar{a}_{k0} - \sum_{j \in N} \left(\bar{a}_{ij} + \delta \bar{a}_{kj}\right) x_j.$$

We then aim to minimize $\sum_{j \in N \cap C} \left(\bar{a}_{ij} + \delta \bar{a}_{kj}\right)^2$ to choose δ, which is a fairly easy task.

Remarks

- Reduce-and-split cuts do not dominate GMI cuts.
- Geometric interpretation: given two splits (π^i, π_0^i), (π^k, π_0^k) defined as in Lemma 2, the reduce-and-split procedure is simply to utilize another split $(\pi^i + \delta\pi^k, \pi_0)$ with $\pi_0 = \lfloor (\pi^i + \delta\pi^k)^\top \bar{x} \rfloor$ to generate GMI cuts.

Consider a mixed-integer linear program with general variable bounds:

$$\begin{aligned} \min_x \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x_i^L \le x \le x_i^U \quad \forall i \in [n] \\ & x_i \in \mathbb{Z} \quad \forall i \in I \end{aligned} \tag{11}$$

where $-\infty \le x_i^L \le x_i^U \le \infty \ \forall i \in [n]$.
Let's write a simplex tableau associated with a basis B:

$$x_i = \bar{a}_{i0} - \sum_{j \in N^L} \bar{a}_{ij} \left(x_j - x_j^L \right) + \sum_{j \in N^U} \bar{a}_{ij} \left(x_j^U - x_j \right) \quad \forall i \in B, \tag{12}$$

where N^L denotes the index set of non-basic variables at their lower bounds and $N^U \equiv N \setminus N^L$.

If $\bar{a}_{i0} \notin \mathbb{Z}$, using the simplex tableau (12), we can derive an GMI inequality as follows:

$$\sum_{j \in N^L \cap I} \min\left\{\frac{f_{ij}}{f_{i0}}, \frac{1-f_{ij}}{1-f_{i0}}\right\}(x_j - x_j^L) + \sum_{j \in N^L \cap C} \max\left\{\frac{-\bar{a}_{ij}}{1-f_{i0}}, \frac{\bar{a}_{ij}}{f_{i0}}\right\}(x_j - x_j^L) + \sum_{j \in N^U \cap I} \min\left\{\frac{f_{ij}}{1-f_{i0}}, \frac{1-f_{ij}}{f_{i0}}\right\}(x_j^U - x_j) + \sum_{j \in N^U \cap C} \max\left\{\frac{\bar{a}_{ij}}{1-f_{i0}}, \frac{-\bar{a}_{ij}}{f_{i0}}\right\}(x_j^U - x_j) \geq 1. \quad (13)$$

Exercise (Cont'd)

Remarks

- As long as x_i^L and x_i^U are valid bounds for x_i, then the resulting GMI inequality (13) will be correct.
- For a superbasic variable j, since $\bar{a}_{ij} = 0$ (due to $B^{-1}a_j = 0$), we can simply skip variable j.
- Given a mixed binary program (i.e., $x_i \in \{0, 1\}$ for $i \in I$), let F_0 and $F_1 \subseteq I$ denote the index sets of variables that have been fixed at 0 and 1, respectively, in some BB node. One can enforce $F_0 \subseteq N^L \cap I$ and $F_1 \subseteq N^U \cap I$ and generate a GMI inequality (13), which will be globally valid inequality [BCCN96]. We cannot extend this to general MILPs.

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