An Introduction to MIR Inequalities and Relations between Families of Cuts

Akang Wang
wangakang@sribd.cn

Shenzhen Research Institute of Big Data

May 18, 2023

Outline

- Gomory mixed-integer (GMI) inequalities/closure
- Mixed-integer rounding (MIR) inequalities/closure
- Relations between families of cuts

Readers are referred to [Cor08, BC08, MW01, DGL10, DGR11, ACL05].

Theorem

Consider a mixed-integer set with a single equality

$$X \equiv \left\{ x \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+^{|C|} : a^\top x = a_0 \right\}.$$

Let $f_j \equiv a_j - \lfloor a_j \rfloor$ *for* $j \in \{0\} \cup I$, *then the GMI inequality*

$$\sum_{j \in I : f_j \leq f_0} \frac{f_j}{f_0} x_j + \sum_{j \in I : f_j > f_0} \frac{1 - f_j}{1 - f_0} x_j + \sum_{j \in C : a_j > 0} \frac{a_j}{f_0} x_j + \sum_{j \in C : a_j < 0} \frac{-a_j}{1 - f_0} x_j \geq 1 \tag{1}$$

is a valid inequality for $conv(X)$.

A proof can be found at this link.

Theorem

Consider a general mixed-integer set

$$X \equiv \left\{ x \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+^{|C|} : a^\top x \leq a_0 \right\}.$$

Let $f_j \equiv a_j - \lfloor a_j \rfloor$ *for* $j \in \{0\} \cup I$, *then the GMI inequality*

$$\sum_{j \in I} \left(\lfloor a_j \rfloor + \frac{(f_j - f_0)^+}{1 - f_0} \right) x_j + \frac{1}{1 - f_0} \sum_{j \in C : a_j < 0} a_j x_j \leq \lfloor a_0 \rfloor \quad (2)$$

is a valid inequality for $conv(X)$.

Proof.

Introduce a slack variable $a^\top x + s = a_0$ and then generate GMI ineq. (1) in the (x, s)-space:

$$\sum_{j\in I: f_j \le f_0} \frac{f_j}{f_0} x_j + \sum_{j\in I: f_j > f_0} \frac{1-f_j}{1-f_0} x_j + \sum_{j\in C: a_j>0} \frac{a_j}{f_0} x_j - \sum_{j\in C: a_j<0} \frac{a_j}{1-f_0} x_j + \frac{1}{f_0} s \ge 1$$

We substitute $s = a_0 - a^\top x$ to get the GMI inequality (2). □

Definition (**GMI closure**)

Consider a general mixed-integer set

$$X \equiv \left\{ x \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+^{|C|} : Ax \leq b \right\},$$

let P denote its continuous relaxation.

i. Add a slack variable $s \in \mathbb{R}_+^m$ and obtain $Ax + s = b$.

ii. For $\lambda \in \mathbb{R}^m$, we have

$$\lambda^\top Ax + \lambda^\top s = \lambda^\top b,$$

iii. Derive a GMI inequality (1) from the above equality, and then eliminate s, resulting a GMI inequality in the x space.

iv. GMI closure relative to P: P^{GMI} is obtained from P by adding all the GMI inequalities.

GMI closure

Definition (GMI closure)

Consider a general mixed-integer set

$$X \equiv \left\{ x \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+^{|C|} : Ax \le b \right\},$$

let P denote its continuous relaxation.

i. Add a slack variable $s \in \mathbb{R}_+^m$ and obtain $Ax + s = b$.

ii. For $\lambda \in \mathbb{R}^m$, we have

$$\lambda^\top Ax + \lambda^\top s = \lambda^\top b,$$

iii. Derive a GMI inequality (1) from the above equality, and then eliminate s, resulting a GMI inequality in the x space.

iv. GMI closure relative to P: P^{GMI} is obtained from P by adding all the GMI inequalities.

Definition (**GMI closure**)

Consider a general mixed-integer set

$$X \equiv \left\{ x \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+^{|C|} : Ax \leq b \right\},$$

let P denote its continuous relaxation.

i. Add a slack variable $s \in \mathbb{R}_+^m$ and obtain $Ax + s = b$.

ii. For $\lambda \in \mathbb{R}^m$, we have

$$\lambda^\top Ax + \lambda^\top s = \lambda^\top b,$$

iii. Derive a GMI inequality (1) from the above equality, and then eliminate s, resulting a GMI inequality in the x space.

iv. GMI closure relative to P: P^{GMI} is obtained from P by adding all the GMI inequalities.

Definition (GMI closure)

Consider a general mixed-integer set

$$X \equiv \left\{ x \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+^{|C|} : Ax \leq b \right\},$$

let P denote its continuous relaxation.

i. Add a slack variable $s \in \mathbb{R}_+^m$ and obtain $Ax + s = b$.

ii. For $\lambda \in \mathbb{R}^m$, we have

$$\lambda^\top Ax + \lambda^\top s = \lambda^\top b,$$

iii. Derive a GMI inequality (1) from the above equality, and then eliminate s, resulting a GMI inequality in the x space.

iv. GMI closure relative to P: P^{GMI} is obtained from P by adding all the GMI inequalities.

Definition (GMI closure)

Consider a general mixed-integer set

$$X \equiv \left\{ x \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+^{|C|} : Ax \leq b \right\},$$

let P denote its continuous relaxation.

i. Add a slack variable $s \in \mathbb{R}_+^m$ and obtain $Ax + s = b$.

ii. For $\lambda \in \mathbb{R}^m$, we have

$$\lambda^\top Ax + \lambda^\top s = \lambda^\top b,$$

iii. Derive a GMI inequality (1) from the above equality, and then eliminate s, resulting a GMI inequality in the x space.

iv. GMI closure relative to P: P^{GMI} is obtained from P by adding all the GMI inequalities.

Remarks

- One may generate GMI inequalities from any valid constraints, however, those inequalities might not cut off the current optimal solution (e.g., basic feasible solution). Hence, in practice, one would generate GMI cuts from the optimal simplex tableau (which is essentially obtained by Gaussian elimination (i.e., linear combination of linear systems), this corresponds to the fact that $\lambda \in \mathbb{R}^m$ in Theorem 2).

Lemma

Consider a 2-variable mixed-integer set

$$X \equiv \left\{ (x, s) \in \mathbb{Z} \times \mathbb{R} : \begin{array}{ll} x \leq a_0 + s & (I) \\ s \geq 0 & (II) \end{array} \right\}.$$

Let $f_0 \equiv a_0 - \lfloor a_0 \rfloor$*, then the inequality*

$$x - \frac{s}{1 - f_0} \leq \lfloor a_0 \rfloor \tag{3}$$

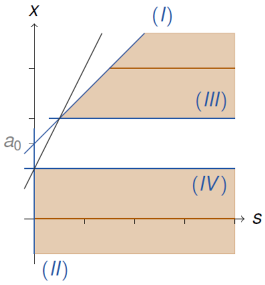
is a valid inequality for $conv(X)$.

Proof.

Consider the following disjunction

$$x \leq \lfloor a_0 \rfloor \ (III) \ \vee \ x \geq \lceil a_0 \rceil \ (IV).$$

Then both $(I) + f_0(III)$ and $(II) + (1 - f_0)(IV)$ will produce the inequality (3). Note that the resulting ineq. is also a split inequality. □



Theorem

Consider a mixed-integer set with a single inequality

$$X \equiv \left\{ x \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+^{|C|} : a^\top x \le a_0 \right\}.$$

Let $f_j \equiv a_j - \lfloor a_j \rfloor$ *for all* $j = \{0\} \cup I$, *then the MIR inequality*

$$\sum_{j \in I} \left(\lfloor a_j \rfloor + \frac{(f_j - f_0)^+}{1 - f_0} \right) x_j + \frac{1}{1 - f_0} \sum_{j \in C : a_j < 0} a_j x_j \le \lfloor a_0 \rfloor \qquad (4)$$

is a valid inequality for $conv(X)$.

Observation

The MIR inequality (4) is identical to the GMI inequality (2).

Theorem

Consider a mixed-integer set with a single inequality

$$X \equiv \left\{ x \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+^{|C|} : a^\top x \le a_0 \right\}.$$

Let $f_j \equiv a_j - \lfloor a_j \rfloor$ *for all* $j = \{0\} \cup I$*, then the MIR inequality*

$$\sum_{j \in I} \left(\lfloor a_j \rfloor + \frac{(f_j - f_0)^+}{1 - f_0} \right) x_j + \frac{1}{1 - f_0} \sum_{j \in C : a_j < 0} a_j x_j \le \lfloor a_0 \rfloor \quad (4)$$

is a valid inequality for $conv(X)$*.*

Observation

The MIR inequality (4) is identical to the GMI inequality (2).

Proof.

Considering that $x \geq 0$, one can relax $a^\top x \leq a_0$ to

$$\sum_{j \in I: f_j \leq f_0} \lfloor a_j \rfloor x_j + \sum_{j \in I: f_j > f_0} a_j x_j + \sum_{j \in C: a_j < 0} a_j x_j \leq a_0.$$

Let

$$w \equiv \sum_{j \in I: f_j \leq f_0} \lfloor a_j \rfloor x_j + \sum_{j \in I: f_j > f_0} \lceil a_j \rceil x_j, \; z \equiv - \sum_{j \in C: a_j < 0} a_j x_j + \sum_{j \in I: f_j > f_0} (1 - f_j) x_j.$$

We have $w - z \leq a_0$. Since $w \in \mathbb{Z}$ and $z \in \mathbb{R}_+$, we can apply Lemma 1. Thus

$$w - \frac{z}{1 - f_0} \leq \lfloor a_0 \rfloor.$$

Substituting w and z yields (4). □

Definition (MIR closure)

Consider a general mixed-integer set

$$X \equiv \left\{ x \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+^{|C|} : Ax \leq b \right\},$$

let P denote its continuous relaxation.

i. By Farkas Lemma, any valid inequalities for P are of the form

$$\lambda^\top Ax - v^\top x \leq \lambda^\top b + t,$$

where $\lambda \in \mathbb{R}_+^m, v \in \mathbb{R}_+^n, t \in \mathbb{R}_+$.

ii. Derive a an MIR inequality (4) from the above inequality.

iii. MIR closure relative to P: P^{MIR} is obtained from P by adding all the MIR inequalities.

Definition (MIR closure)

Consider a general mixed-integer set

$$X \equiv \left\{ x \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+^{|C|} : Ax \leq b \right\},$$

let P denote its continuous relaxation.

i. By Farkas Lemma, any valid inequalities for P are of the form

$$\lambda^\top Ax - v^\top x \leq \lambda^\top b + t,$$

where $\lambda \in \mathbb{R}_+^m, v \in \mathbb{R}_+^n, t \in \mathbb{R}_+$.

ii. Derive a an MIR inequality (4) from the above inequality.

iii. MIR closure relative to P: P^{MIR} is obtained from P by adding all the MIR inequalities.

Definition (MIR closure)

Consider a general mixed-integer set

$$X \equiv \left\{x \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+^{|C|} : Ax \leq b\right\},$$

let P denote its continuous relaxation.

i. By Farkas Lemma, any valid inequalities for P are of the form

$$\lambda^\top Ax - v^\top x \leq \lambda^\top b + t,$$

where $\lambda \in \mathbb{R}_+^m, v \in \mathbb{R}_+^n, t \in \mathbb{R}_+$.

ii. Derive a an MIR inequality (4) from the above inequality.

iii. MIR closure relative to P: P^{MIR} is obtained from P by adding all the MIR inequalities.

Definition (MIR closure)

Consider a general mixed-integer set

$$X \equiv \left\{ x \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+^{|C|} : Ax \leq b \right\},$$

let P denote its continuous relaxation.

i. By Farkas Lemma, any valid inequalities for P are of the form

$$\lambda^\top A x - v^\top x \leq \lambda^\top b + t,$$

where $\lambda \in \mathbb{R}_+^m, v \in \mathbb{R}_+^n, t \in \mathbb{R}_+$.

ii. Derive a an MIR inequality (4) from the above inequality.

iii. MIR closure relative to P: P^{MIR} is obtained from P by adding all the MIR inequalities.

Complemented MIR (c-MIR)

Theorem (c-MIR)

Consider the following mixed-integer knapsack set

$$X \equiv \left\{(x,s) \in \mathbb{Z}_+^n \times \mathbb{R}_+ : a^\top x - s \le a_0, x_i \le u_i \ \forall i \in [n]\right\}$$

Let T and U be a partition of set $\{1,2,...,n\}$, $\delta > 0$

$$\sum_{i\in T} G\left(\frac{a_i}{\delta}\right) x_i + \sum_{i \in U} G\left(\frac{-a_i}{\delta}\right)(u_i - x_i) - \frac{s}{\delta(1-f_0)} \le \lfloor \beta \rfloor,$$

where $\beta \equiv \frac{a_0 - \sum_{j\in U} a_j u_j}{\delta}$, $f_0 \equiv \beta - \lfloor\beta\rfloor$, $G(y) \equiv \lfloor y \rfloor + \frac{(f_y - f_0)^+}{1-f_0}$ *with* $f_y \equiv y - \lfloor y \rfloor$.

Proof.

Using Eq. (4), it is easy to demonstrate the validity. □

Complemented MIR (c-MIR)

Theorem (c-MIR)

Consider the following mixed-integer knapsack set

$$X \equiv \left\{(x,s) \in \mathbb{Z}_+^n \times \mathbb{R}_+ : a^\top x - s \leq a_0, x_i \leq u_i \ \forall i \in [n]\right\}$$

Let T and U be a partition of set $\{1,2,\ldots,n\}$, $\delta > 0$

$$\sum_{i\in T} G\left(\frac{a_i}{\delta}\right) x_i + \sum_{i\in U} G\left(\frac{-a_i}{\delta}\right)(u_i - x_i) - \frac{s}{\delta(1-f_0)} \leq \lfloor\beta\rfloor,$$

where $\beta \equiv \frac{a_0 - \sum_{j\in U} a_j u_j}{\delta}$, $f_0 \equiv \beta - \lfloor\beta\rfloor$, $G(y) \equiv \lfloor y\rfloor + \frac{(f_y - f_0)^+}{1-f_0}$ with $f_y \equiv y - \lfloor y\rfloor$.

Proof.

Using Eq. (4), it is easy to demonstrate the validity. □

Three steps: (i) aggregation; (ii) bound substitution; (iii) separation.

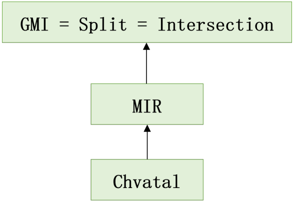
Algorithm 8.2 Separation of Complemented MIR Cuts

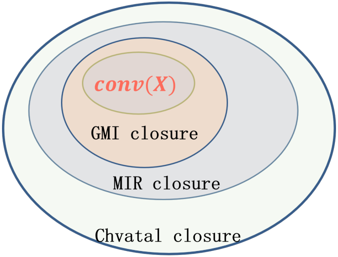
Input: LP relaxation $Ax \le b$ of MIP, global bounds $l \le x \le u$, and current LP solution $\check{x}$.

Output: Cutting planes $d^T x \le \bar{\gamma}$.

1. Aggregate linear constraints to obtain a single linear inequality $a^T x \le \bar{\beta}$.
2. Transform the variables to the canonical form $x' \ge 0$ by either
   - ▷ shifting to their lower bound $x_j \ge l_j$: $x'_j := x_j - l_j$,
   - ▷ complementing to their upper bound $x_j \le u_j$: $x'_j := u_j - x_j$,
   - ▷ substituting with a variable lower bound $x_j \ge s x_k + d$, $k \in I$: $x'_j := x_j - (s x_k + d)$, or
   - ▷ substituting with a variable upper bound $x_j \le s x_k + d$, $k \in I$: $x'_j := (s x_k + d) - x_j$.
3. Divide the resulting inequality $a'^T x' \le \bar{\beta}'$ by $\delta = \pm 1$, $\delta = \pm \max\{|a'_j| \mid j \in I\}$, and $\delta \in \{\pm a'_j \mid j \in I \text{ and } 0 < \check{x}'_j < u'_j\}$, and generate the corresponding MIR inequalities (8.3). Choose $\delta^\star$ to be the δ for which the most violated MIR inequality has been produced.
4. In addition to $\delta^\star$, check whether the MIR inequalities derived from dividing $a'^T x' \le \bar{\beta}'$ by $\frac{1}{2}\delta^\star$, $\frac{1}{4}\delta^\star$, and $\frac{1}{8}\delta^\star$ yield even larger violations.
5. Finally, select the most violated of the MIR inequalities, transform it back into the space of problem variables x, substitute slack variables, and add it to the separation storage.

Readers are referred to [MW01, Ach07] for more details.





Remarks

- $X^{GMI} \subsetneq X^{MIR}$ does not imply that MIR separation is futile when GMI inequalities are separated, since only a subset of GMI inequalities (rather than all inequalities that define the GMI closure) are added in practice.

Theorem (GMI closure vs MIR closure)

Consider a general mixed-integer set

$$X \equiv \left\{ x \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+^{|C|} : Ax \leq b \right\},$$

let P denote its continuous relaxation. Then $X^{GMI} \subsetneq X^{MIR}$.

Remarks

- As [Cor08, BC08, DGL10] pointed out, $X^{GMI} \subsetneq X^{MIR}$ results from the sign of λ: $\lambda \in \mathbb{R}^m$ for X^{GMI}, $\lambda \in \mathbb{R}_+^m$ for X^{MIR}.

Theorem (GMI closure vs MIR closure)

Consider a general mixed-integer set

$$X \equiv \left\{ x \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+^{|C|} : Ax \leq b \right\},$$

let P denote its continuous relaxation. Then $X^{GMI} \subsetneq X^{MIR}$.

Remarks

- As [Cor08, BC08, DGL10] pointed out, $X^{GMI} \subsetneq X^{MIR}$ results from the sign of λ: $\lambda \in \mathbb{R}^m$ for X^{GMI}, $\lambda \in \mathbb{R}_+^m$ for X^{MIR}.

Chvatal vs MIR/GMI

深圳市大数据研究院
Shenzhen Research Institute of Big Data

Lemma

Consider a general mixed-integer set

$$X \equiv \left\{x \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+^{|C|} : Ax \leq b\right\},$$

let P denote its continuous relaxation. Any Chvatal inequalities are GMI inequalities.

Proof.

Given a Chvatal inequality $\pi^\top x \leq \pi_0$ with $\pi_I \in \mathbb{Z}^{|I|}, \pi_C = 0$ and $\pi_0 \in \mathbb{Z}$, we know $P \cap \left\{x \in \mathbb{R}^n : \pi^\top x \geq \pi_0 + 1\right\} = \emptyset$. Let $\beta \equiv \max_{x \in P} \pi^\top x$, then $\pi_0 \leq \beta < \pi_0 + 1$. Derive an MIR inequality (4) from $\pi^\top x \leq \beta$, we obtain $\pi^\top x \leq \pi_0$. By Observation 10, this inequality is also a GMI inequality. □

Chvatal vs MIR/GMI

Lemma

Consider a general mixed-integer set

$$X \equiv \left\{ x \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+^{|C|} : Ax \leq b \right\},$$

let P denote its continuous relaxation. Any Chvatal inequalities are GMI inequalities.

Proof.

Given a Chvatal inequality $\pi^\top x \leq \pi_0$ with $\pi_I \in \mathbb{Z}^{|I|}, \pi_C = 0$ and $\pi_0 \in \mathbb{Z}$, we know $P \cap \left\{ x \in \mathbb{R}^n : \pi^\top x \geq \pi_0 + 1 \right\} = \emptyset$. Let $\beta \equiv \max_{x \in P} \pi^\top x$, then $\pi_0 \leq \beta < \pi_0 + 1$. Derive an MIR inequality (4) from $\pi^\top x \leq \beta$, we obtain $\pi^\top x \leq \pi_0$. By Observation 10, this inequality is also a GMI inequality. □

Definition (Chvatal closure)

Consider a general mixed-integer set

$$X \equiv \left\{ x \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+^{|C|} : Ax \leq b \right\},$$

let P denote its continuous relaxation. Then the Chvatal closure relative to P is defined as

$$X^{Chvatal} \equiv \bigcap_{\substack{\pi_I \in \mathbb{Z}^{|I|}, \pi_C = 0, \pi_0 \in \mathbb{Z}: \\ \left\{ x \in P : \pi^\top x \geq \pi_0 + 1 \right\} = \emptyset}} \left\{ x \in P : \pi^\top x \leq \pi_0 \right\}$$

Definition (Split closure)

Consider a general mixed-integer set

$$X \equiv \left\{ x \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+^{|C|} : Ax \leq b \right\},$$

let P denote its continuous relaxation. Then the split closure relative to P is defined as

$$X^{split} \equiv \bigcap_{\substack{\pi_I \in \mathbb{Z}^{|I|}, \pi_C = 0 \\ \pi_0 \in \mathbb{Z}}} conv \left(\left\{ x \in P : \pi^\top x \leq \pi_0 \right\} \cup \left\{ x \in P : \pi^\top x \geq \pi_0 + 1 \right\} \right)$$

GMI = Split

In order to show the equivalence between split inequalities and GMI cuts, we need the following lemma.

Lemma

Given a polyhedron $P \equiv \{x \in \mathbb{R}^n : Ax \leq b\}$, *let* $\Pi \equiv \{x \in P : \pi^\top x \leq \pi_0\}$. *If* $\Pi \neq \emptyset$ *and* $\alpha^\top x \leq \beta$ *is a valid inequality for* Π, *then there exists a scalar* $\lambda \in \mathbb{R}_+$ *such that* $\alpha^\top x - \lambda\left(\pi^\top x - \pi_0\right) \leq \beta$ *is valid for* P.

Proof.

Since $\Pi \neq \emptyset$, by Farkas Lemma, there exists $u \in \mathbb{R}^m_+, \lambda \in \mathbb{R}_+$ such that

$$\alpha = A^\top u + \lambda\pi \text{ and } \beta \geq u^\top b + \lambda\pi_0.$$

Note that $P \neq \emptyset$ (since $\Pi \neq \emptyset$), hence $u^\top Ax \leq u^\top b$ is valid for P. As a result, $\left(\alpha^\top - \lambda\pi^\top\right) x \leq \beta - \lambda\pi_0$. Rearrange this inequality, we can come to the conclusion. □

In order to show the equivalence between split inequalities and GMI cuts, we need the following lemma.

Lemma

Given a polyhedron $P \equiv \{x \in \mathbb{R}^n : Ax \leq b\}$, let $\Pi \equiv \left\{x \in P : \pi^\top x \leq \pi_0\right\}$. If $\Pi \neq \emptyset$ and $\alpha^\top x \leq \beta$ is a valid inequality for Π, then there exists a scalar $\lambda \in \mathbb{R}_+$ such that $\alpha^\top x - \lambda\left(\pi^\top x - \pi_0\right) \leq \beta$ is valid for P.

Proof.

Since $\Pi \neq \emptyset$, by Farkas Lemma, there exists $u \in \mathbb{R}^m_+, \lambda \in \mathbb{R}_+$ such that

$$\alpha = A^\top u + \lambda\pi \text{ and } \beta \geq u^\top b + \lambda\pi_0.$$

Note that $P \neq \emptyset$ (since $\Pi \neq \emptyset$), hence $u^\top Ax \leq u^\top b$ is valid for P. As a result, $\left(\alpha^\top - \lambda\pi^\top\right) x \leq \beta - \lambda\pi_0$. Rearrange this inequality, we can come to the conclusion. □

GMI = Split

深圳市大数据研究院
Shenzhen Research Institute of Big Data

Theorem (GMI = Split)

Consider a general mixed-integer set

$$X \equiv \left\{ x \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+^{|C|} : Ax \leq b \right\},$$

let P denote its continuous relaxation. Then $X^{split} = P^{GMI}$.

Proof.

In our previous talk, we have shown "⇐"; next we show "⇒". Given a split (π, π_0), let $\alpha^\top x \leq \beta$ be a valid inequality for both sets

$$\Pi_1 \equiv \left\{ x \in P : \pi^\top x \leq \pi_0 \right\},$$

$$\Pi_2 \equiv \left\{ x \in P : \pi^\top x \geq \pi_0 + 1 \right\}.$$

GMI = Split

深圳市大数据研究院
Shenzhen Research Institute of Big Data

Theorem (GMI = Split)

Consider a general mixed-integer set

$$X \equiv \left\{ x \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+^{|C|} : Ax \le b \right\},$$

let P denote its continuous relaxation. Then $X^{split} = P^{GMI}$.

Proof.

In our previous talk, we have shown "$\Leftarrow$"; next we show "$\Rightarrow$". Given a split (π, π_0), let $\alpha^\top x \le \beta$ be a valid inequality for both sets

$$\Pi_1 \equiv \left\{ x \in P : \pi^\top x \le \pi_0 \right\},$$
$$\Pi_2 \equiv \left\{ x \in P : \pi^\top x \ge \pi_0 + 1 \right\}.$$

Proof (Cont'd).

Case I: If $\Pi_2 = \emptyset$, since all the inequalities that define Π_1 are valid for P except possibly for $\pi^\top x \le \pi_0$, it suffices to show that $\pi^\top x \le \pi_0$ is a GMI inequality. This follows from the fact that $\pi^\top x \le \pi_0$ is a Chvatal inequality and from Lemma 8.

Case II: If $\Pi_1 \neq \emptyset$ and $\Pi_2 \neq \emptyset$, by Lemma 11, there exists $u, v \in \mathbb{R}_+$ such that inequalities (5) and (6) are valid for P.

$$\alpha^\top x - u\left(\pi^\top x - \pi_0\right) \le \beta \tag{5}$$

$$\alpha^\top x - v\left(-\pi^\top x + \pi_0 + 1\right) \le \beta \tag{6}$$

Introduce slack variables $s_1, s_2 \in \mathbb{R}_+$ and subtract (5) from (6), we obtain

$$(v+u)\pi^\top x + s_2 - s_1 = (v+u)\pi_0 + v.$$

Proof (Cont'd).

Divide by $(v+u)$,

$$\pi^\top x + \frac{s_2}{v+u} - \frac{s_1}{v+u} = \pi_0 + \frac{v}{u+v}. \quad (7)$$

Using (7) as a base equality, we generate a GMI inequality

$$\frac{\frac{1}{v+u}}{\frac{v}{v+u}} s_2 + \frac{\frac{1}{v+u}}{1-\frac{v}{v+u}} s_1 \geq 1,$$

$$\frac{1}{v} s_2 + \frac{1}{u} s_1 \geq 1.$$

Substitute s_1 and s_2, we have

$$\alpha^\top x \leq \beta.$$

□

Theorem

Consider a general mixed-integer set

$$X \equiv \left\{ x \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+^{|C|} : Ax = b \right\}.$$

WLOG, assume that matrix A is of full row rank. Let B index m linearly independent columns of A (basic) and let N index the remain columns of A (nonbasic). Associated with B is a corner polyhedron

$$P(B) \equiv \left\{ x \in \mathbb{R}_+^n : Ax = b, x_j \geq 0 \ \forall j \in N \right\}.$$

We claim that

$$X^{split} = \bigcap_{B \in \mathcal{B}} \bigcap_{\substack{\pi_I \in \mathbb{Z}^{|I|}, \pi_C = 0 \\ \pi_0 \in \mathbb{Z}}} conv\left(P(B) \cap \left(\left\{x : \pi^\top x \leq \pi_0\right\} \cup \left\{x : \pi^\top x \geq \pi_0 + 1\right\}\right)\right)$$

$$= \bigcap_{B \in \mathcal{B}} \bigcap_{\substack{\pi_I \in \mathbb{Z}^{|I|}, \pi_C = 0 \\ \pi_0 \in \mathbb{Z}}} \left\{x \in P(B) : \sum_{j \in N} x_j / \alpha_j \leq \pi_0\right\}$$

References I

[Ach07] Tobias Achterberg, *Constraint integer programming*.

[ACL05] Kent Andersen, Gérard Cornuéjols, and Yanjun Li, *Split closure and intersection cuts*, Mathematical programming **102** (2005), no. 3, 457–493.

[BC08] Pierre Bonami and Gérard Cornuéjols, *A note on the mir closure*, Operations research letters **36** (2008), no. 1, 4–6.

[Cor08] Gérard Cornuéjols, *Valid inequalities for mixed integer linear programs*, Mathematical programming **112** (2008), no. 1, 3–44.

[DGL10] Sanjeeb Dash, Oktay Günlük, and Andrea Lodi, *Mir closures of polyhedral sets*, Mathematical Programming **121** (2010), no. 1, 33–60.

References II

[DGR11] Sanjeeb Dash, Oktay Günlük, and Christian Raack, *A note on the mir closure and basic relaxations of polyhedra*, Operations Research Letters **39** (2011), no. 3, 198–199.

[MW01] Hugues Marchand and Laurence A Wolsey, *Aggregation and mixed integer rounding to solve mips*, Operations research **49** (2001), no. 3, 363–371.