An Introduction to Chvatal-Gomory Cuts

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Outline

- Chvatal-Gomory (CG) cuts [Chv73]
- Three variants of CG cuts
  - strong CG cuts [LL02]
  - 0-1/2 cuts [CF96]
  - mod-k cuts [CFL00]

Consider an integer program:

$$\begin{aligned} \min_{x} \quad & c^\top x \\ \text{s.t.} \quad & x \in X, \end{aligned} \tag{1}$$

where $X \equiv \{x \in P : x_i \in \mathbb{Z}\ \forall i \in I\}$, $P \equiv \left\{x \in \mathbb{R}^n_+ : Ax \leq b\right\}$, $I = [n]$.
Let $conv(X)$ denote the convex hull of X.

Chvatal-Gomory cuts

Given $X \equiv \{x \in \mathbb{Z}^n_+ : Ax \leq b\}$ and any $u \in \mathbb{R}^m_+$, we have

$$u^\top Ax \leq u^\top b$$
$$\lfloor u^\top Ax \rfloor \leq \lfloor u^\top b \rfloor$$

Since $x_j \in \mathbb{Z}_+$, therefore,

$$\sum_{j \in I} \lfloor u^\top A_{.j} \rfloor x_j \leq \lfloor u^\top b \rfloor \tag{2}$$

The resulting inequality (2) is called a "Chvatal-Gomory cut".

Remarks

- CG cuts depend on on P, not directly on $conv(X)$.
- If $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, then undominated CG cuts only arise for $u \in [0, 1)^m$.

Theorem ([CL01])

Given $X \equiv \left\{x \in \mathbb{Z}^n_+ : Ax \leq b\right\}$, *define Chvatal closure*

$$X^{Chvatal} \equiv \left\{x \in \mathbb{R}^n : \lfloor u^\top A \rfloor x \leq \lfloor u^\top b \rfloor \quad \forall u \in \mathbb{R}^m_+\right\}$$

and Gomory fractional closure

$$X^F \equiv \left\{x \in \mathbb{R}^n : \lfloor \lambda^\top A \rfloor x - \lfloor \lambda^\top \rfloor Ax \leq \lfloor \lambda^\top b \rfloor - \lfloor \lambda^\top \rfloor b \quad \forall \lambda \in \mathbb{R}^m\right\}$$

Then $X^{Chvatal} = X^F$

Check Gomory fractional cuts.

Proof.

- WLOG, assume $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$, hence we consider a CG cut $\lfloor u^\top A \rfloor x \le \lfloor u^\top b \rfloor$ with $u \in [0,1)$. As a result,

$$\lfloor u^\top A \rfloor x - \lfloor u^\top \rfloor A x \le \lfloor u^\top b \rfloor - \lfloor u^\top \rfloor b,$$

  which is a Gomory fractional cut.
- Given a fractional cut $\lfloor \lambda^\top A \rfloor x - \lfloor \lambda^\top \rfloor A x \le \lfloor \lambda^\top b \rfloor - \lfloor \lambda^\top \rfloor b$, one has

$$\left\lfloor \lambda^\top A - \lfloor \lambda^\top \rfloor A \right\rfloor x \le \left\lfloor \lambda^\top b - \lfloor \lambda^\top \rfloor b \right\rfloor.$$

  This is a Chvatal cut $\lfloor u^\top A \rfloor x \le \lfloor u^\top \rfloor b$ with $u \equiv \lambda - \lfloor \lambda \rfloor$.

□

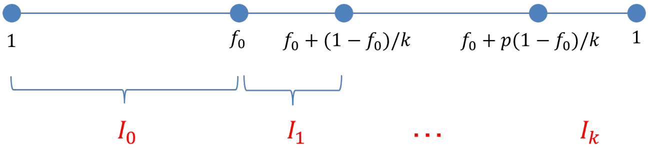
Theorem ([LL02])

Given $X \equiv \{x \in \mathbb{Z}_+^{|I|} : Ax \le b\}$, *a valid inequality* $u^\top Ax \le u^\top b$ *with* $u \in \mathbb{R}_+^m$ *can be written as*

$$\sum_{j \in I} a_j x_j \le a_0. \tag{3}$$

Let $f_j \equiv a_j - \lfloor a_j \rfloor$ *for* $j \in I \cup \{0\}$ *and* $k \ge 1$ *be an integer such that* $\frac{1}{1+k} \le f_0 < \frac{1}{k}$. *Partition* I *into classes* $I_0, I_1, \ldots, I_k$ *as follows. Let* $I_0 \equiv \{j \in I : f_j \le f_0\}$, *and for* $p = 1, 2, \ldots, k$, *let* $I_p \equiv \left\{ j \in I : f_0 + \frac{(p-1)(1-f_0)}{k} < f_j \le f_0 + \frac{p(1-f_0)}{k} \right\}$. *The inequality (4) is valid for* S *and dominates the CG cut (2).*

$$\sum_{p=0}^{k} \sum_{j \in I_p} \left(\lfloor a_j \rfloor + \frac{p}{1+k} \right) x_j \le \lfloor a_0 \rfloor. \tag{4}$$


1
f_0
$f_0 + (1 - f_0)/k$
$f_0 + p(1 - f_0)/k$
1
I_0
I_1
...
I_k

Proof.

After multiplying (3) by k and applying a Chvatal procedure, we obtain

$$\sum_{j \in I} \lfloor k a_j \rfloor x_j \leq \lfloor k a_0 \rfloor. \tag{5}$$

Note that $f_0 = a_0 - \lfloor a_0 \rfloor < \frac{1}{k}$, hence $k a_0 - k \lfloor a_0 \rfloor < 1$. Furthermore, $k a_0 \geq \lfloor k a_0 \rfloor \geq k \lfloor a_0 \rfloor$, therefore, $\lfloor k a_0 \rfloor = k \lfloor a_0 \rfloor$.
For $j \in I$ such that $\frac{p}{k} \leq f_j < \frac{p+1}{k}$, $k \lfloor a_j \rfloor + p \leq k a_j$, hence
$k \lfloor a_j \rfloor + p \leq \lfloor k a_j \rfloor$.

$$\sum_{p=0}^{k-1} \sum_{\substack{j \in I: \\ \frac{p}{k} \leq f_j < \frac{p+1}{k}}} \left(k \lfloor a_j \rfloor + p \right) x_j \leq k \lfloor a_0 \rfloor. \tag{6}$$

Proof (Cont'd).

Choose $\delta > 0$ such that $1 - \delta \geq \frac{f_0}{f_j}$ for all $j \in I \setminus I_0$. We can multiply (3) by $\frac{1-\delta}{f_0}$ and multiply (6) by $1 + \frac{1 - \frac{1-\delta}{f_0}}{k}$, sum the two resulting inequalities together to obtain the valid inequality

$$\sum_{p=0}^{k-1} \sum_{\substack{j \in I: \\ \frac{p}{k} \leq f_j < \frac{p+1}{k}}} \left((k+1)\lfloor a_j \rfloor + p + \frac{(1-\delta)(f_j - \frac{p}{k})}{f_0} + \frac{p}{k} \right) x_j \leq (k+1)\lfloor a_0 \rfloor + 1 - \delta. \quad (7)$$

Applying integer rounding to (7) gives the strong CG inequality (4). □

Strong CG cuts

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Remark

- There is no dominance between strong CG cuts and MIR cuts.

$$\sum_{p=0}^{k} \sum_{j \in I_p} \left(\lfloor a_j \rfloor + \frac{p}{1+k} \right) x_j \leq \lfloor a_0 \rfloor \quad \text{(strong CG)}$$

$$\sum_{j \in I} \left(\lfloor a_j \rfloor + \frac{(f_j - f_0)^+}{1 - f_0} \right) x_j \leq \lfloor a_0 \rfloor \quad \text{(MIR)}$$

0-1/2 cuts

For a pure integer program with $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$, given a fractional solution x^*, we aim to solve

$$\max_{u \in \{0, \frac{1}{2}\}^m} \quad \lfloor u^\top A \rfloor x^* - \lfloor u^\top b \rfloor \tag{8}$$

Remark

- Problem (8) is $\mathcal{NP}$-hard [CF96].
- Neither the strengthening procedure (4) nor the MIR strengthening technique is applicable to 0-1/2 cuts.

Separation protocol [KZK07]

i. reduction

ii. heuristic separation: enumerate all possible combinations of k rows

0-1/2 cuts: reduction

Define $s = b - Ax^*$, $\bar{A} \equiv A \bmod 2$, and $\bar{b} \equiv b \bmod 2$, we can process $(\bar{A}, \bar{b})$ as follows:

i. All columns in $\bar{A}$ corresponding to $x_i^* = 0$ can be removed

ii. Zero rows in $(\bar{A}, \bar{b})$ can be removed

iii. Zero columns in $\bar{A}$ can be removed

iv. Identical columns in $\bar{A}$ can be replaced by a single representative with associated variable value as a sum of the merged variables

v. Any unit vector columns $\bar{A}_{\cdot i} = e_j$ in $\bar{A}$ can be removed provided that x_i^* is added to the slack s_j of row j

vi. Any row j with slack $s_j \geq 1$ can be removed (see next slide)

vii. Identical rows in $(\bar{A}, \bar{b})$ can be removed except for the one with smallest slack value.

Lemma

There exists a vector $u \in \{0, \frac{1}{2}\}^m$ *such that* $\lfloor u^\top A \rfloor x^* - \lfloor u^\top b \rfloor > 0$ *iff there exists a vector* $v \in \{0, 1\}^m$ *such that*

$$\begin{aligned} v^\top \bar{b} \bmod 2 &= 1, \\ v^\top s + (v^\top \bar{A} \bmod 2) x^* &< 1. \end{aligned}$$

Proof.

$$\begin{aligned} \lfloor u^\top A \rfloor x^* - \lfloor u^\top b \rfloor &= \frac{1}{2}\left((2u)^\top b \bmod 2\right) - u^\top s - \frac{1}{2}\left((2u)^\top A \bmod 2\right) x^* \\ &= \frac{1}{2}\left(\left(v^\top \bar{b} \bmod 2\right) - v^\top s - \left(v^\top \bar{A} \bmod 2\right) x^*\right) \end{aligned}$$

□

Mod-k cuts

For a pure integer program with $A \in \mathbb{Z}^{m\times n}$ and $b \in \mathbb{Z}^m$, we aim to generate CG cuts with weights $u \in \left\{0, \frac{1}{k}, \ldots, \frac{k-1}{k}\right\}^m$.

- The $0-1/2$ cut is a special case of mod-k cuts.
- Consider the following separation problem

$$\begin{aligned} \max_{u\in\left\{0,\frac{1}{k},\ldots,\frac{k-1}{k}\right\}^m} \quad & u^\top Ax^* - \lfloor u^\top b \rfloor \\ \text{s.t.} \quad & u^\top A \in \mathbb{Z}^n \end{aligned}$$

  This problem is $\mathcal{NP}$-hard, however, given k, finding a maximally violated mod-k cut (vio $= (k-1)/k$) can be achieved in $\mathcal{O}(mn\min\{m,n\})$ [CFL00].
- In practice, this type of cut is often separated for combinatorial optimization problems such as TSP.

[CF96] Alberto Caprara and Matteo Fischetti, *{0, 1/2}-chvátal-gomory cuts*, Mathematical Programming **74** (1996), no. 3, 221–235.

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[CL01] Gérard Cornuéjols and Yanjun Li, *Elementary closures for integer programs*, Operations Research Letters **28** (2001), no. 1, 1–8.

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References II

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