An Introduction to Cover Inequalities: Part I

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Knapsack Polytope

Consider a knapsack set:

$$K \equiv \left\{ x \in \{0,1\}^n : a^\top x \le b \right\}, \tag{1}$$

where $0 < a_i \le b$ for all $i \in [n]$ and $\sum_{i=1}^{n} a_i > b$.

Remark

- If $a_i < 0$, we can complement variable x_i; if $a_i > b$, we set $x_i = 0$.
- Applications: resource allocation, vehicle routing, etc.

Let $P \equiv conv(K)$, we call P a *knapsack polytope*. We are interested in characterizing P.

Observation

$dim(P) = n$. (e.g., $n+1$ affinely independent points e^i for $i \in [n]$ and 0)

Definition (Affinely independent)

Vectors $x^1, x^2, ..., x^m$ are affinely independent if $\lambda_i = 0$ for all $i \in [m]$ is a unique solution to the system

$$\sum_{i=1}^{m} \lambda_i x^i = 0, \sum_{i=1}^{m} \lambda_i = 0$$

Definition (Affine Dimension)

The dimension of a set $S \subseteq \mathbb{R}^n$, denoted by $dim(S)$, is the maximum number of affinely independent points in S minus one.

Definition (Cover Inequalities)

A set $C \subseteq [n]$ is called a cover w.r.t. K if $a(C) \equiv \sum_{i \in C} a_i > b$. It is called minimal if it does not contain a proper subset that is a cover, i.e., $C \setminus \{i\}$ is not a cover for any $i \in C$. The corresponding (minimal) cover inequality (CI)

$$\sum_{i \in C} x_i \leq |C| - 1. \tag{2}$$

is valid for K.

Remark

In general, ineq. (2) is not guaranteed to define a facet of $P^{a,b}$.

Proposition

Let C denote a minimal cover w.r.t. K, then (2) defines a facet of the knapsack polytope

$$P_C \equiv conv\left\{x \in \{0,1\}^C : \sum_{i \in C} a_i x_i \leq b\right\},$$

and it defines a face of P of dimension at least $|C| - 1$.

Proof.

Since (2) is valid for P_C, hence

$$P_C \cap \left\{ x \in \mathbb{R}^C : \sum_{i \in C} x_i = |C| - 1 \right\}$$

defines a non-empty face. Since the set of points $(\mathbb{1} - e^i)$ for $i \in C$ are affinely independent, thus (2) defines a facet of P_C.

Clearly, (2) defines a face of P:

$$P \cap \left\{ x \in \mathbb{R}^n : \sum_{i \in C} x_i = |C| - 1 \right\},$$

The dimension of this face is at least $|C| - 1$. □

Definition (Extended Cover Inequalities)

Let $a^* \equiv \max_{i \in C} a_i$ and define the extension of C as $E(C) \equiv C \cup \{i \in [n] \setminus C : a_i \geq a^*\}$, the extended cover inequality (ECI)

$$\sum_{i \in E(C)} x_i \leq |C| - 1 \quad (3)$$

is valid for K.

Remark

- (3) dominates (2).
- In general, ECIs are not guaranteed to define facets of P.

Remark

If $\mathcal{C}$ is the family of all minimal covers C w.r.t. K, then

$$K = \left\{ x \in \{0,1\}^n : \sum_{i \in C} x_i \le |C| - 1 \quad \forall C \in \mathcal{C} \right\},$$

$$K = \left\{ x \in \{0,1\}^n : \sum_{i \in E(C)} x_i \le |C| - 1 \quad \forall C \in \mathcal{C} \right\}.$$

Definition (Lifted Cover Inequalities)

Given a partition of a minimal cover C into two disjoint sets C_1 and C_2 with $C_1 \neq \emptyset$, we consider some $\hat{\alpha}_i \geq 0, \tilde{\alpha}_i \geq 0$ such that an equality given by (4) is valid for K.

$$\sum_{i \in C_1} x_i + \sum_{i \in [n] \setminus C} \hat{\alpha}_i x_i + \sum_{i \in C_2} \tilde{\alpha}_i x_i \leq |C_1| - 1 + \sum_{i \in C_2} \tilde{\alpha}_i \tag{4}$$

(4) is called “Lifted Cover Inequality” (LCI).

Remark

- If $C_2 = \emptyset$, it is often called “simple LCI”; otherwise, “general LCI”.
- Maximally lifting up all variables in $[n] \setminus C$, i.e., making $\hat{\alpha}_i$ as large as possible, and maximally lifting down all variables in C_2, i.e., making $\tilde{\alpha}_i$ as small as possible.

Up-Lifting / Down-Lifting

Take an inequality $\sum_{i \in S} \alpha_i x_i \leq \beta$ that is valid for $\{x \in K : x_i = 0 \ \forall i \in S\}$ / $\{x \in K : x_i = 1 \ \forall i \in S\}$ and turn it into a valid inequality for K.

- sequential lifting
- simultaneous lifting (a.k.a. superadditive lifting)

Sequential Up-Lifting

Proposition (Sequential Up-Lifting, [Gu95])

Suppose $K \subseteq \{0,1\}^n$, $K^d \equiv \{x \in K : x_1 = d\}$ *where* $d \in \{0,1\}$. *Suppose*

$$\sum_{i=2}^{n} \alpha_i x_i \leq \beta \tag{5}$$

is valid for K^0. *If* $K^1 \neq \emptyset$, *then*

$$\alpha_1 x_1 + \sum_{i=2}^{n} \alpha_i x_i \leq \beta \tag{6}$$

is valid for K, *where* $\alpha_1 \equiv \beta - \xi$ *and* $\xi \equiv \max_{x \in K^1} \sum_{i=2}^{n} \alpha_i x_i$. *Moreover, if* K^0 *is full-dimensional and (5) defines a facet of* $\text{conv}(K^0)$, *then* K *is full-dimensional and (6) defines a facet of* K.

Proof.

The validity of (6) is obvious.
If K^0 is full-dimensional and (5) defines a facet of $conv(K^0)$, we consider $|C|$ affinely independent points

$$\begin{gathered}(0,1,1,...,1,0),\\(0,1,1,...,0,1),\\\vdots\\(0,0,1,...,1,1).\end{gathered}$$

Now we consider a new point

$$(1,0,...,1,0,1).$$

Together, there are $|C|+1$ affinely independent points (i.e., dimension increases by 1). □

Sequential Up-Lifting

To find ξ, we consider the following subproblem:

$$\begin{aligned} \max_{x \in \{0,1\}^n} \quad & \sum_{i=2}^{n} \alpha_i x_i \\ \text{s.t.} \quad & \sum_{i \in [n]} a_i x_i \leq b \\ & x_1 = 1 \end{aligned} \tag{7}$$

Remark

- If $\alpha_i \in \mathbb{Z}$, then $\xi \in \mathbb{Z}$.
- When applying the sequential up-lifting to tighten CIs, (7) can be solved by *dynamic programming* in $\mathcal{O}(n|C|)$.
- All coefficient lifting can be computed in $\mathcal{O}(n^2|C|)$ time. This can be further reduced to $\mathcal{O}(n|C|)$ [Zem89].

Example

Consider a knapsack inequality $4x_1 + 5x_2 + 6x_3 + 7x_4 + 9x_5 \leq 13$. The CI associated with a minimal cover $C \equiv \{2, 3, 4\}$ is $x_2 + x_3 + x_4 \leq 2$. Lifting this inequality, we have two choices in ordering $[n] \setminus C \equiv \{1, 5\}$.
Lifting the variable x_1:

$$\begin{aligned} \alpha_1 &\equiv 2 - \max_{x \in \{0,1\}^C} \left\{ \sum_{i \in C} x_i : \sum_{i \in C} a_i x_i \leq \beta - a_1 \right\} \\ &= 2 - \max_{x \in \{0,1\}^C} \{x_2 + x_3 + x_4 : 5x_2 + 6x_3 + 7x_4 \leq 13 - 4\} \\ &= 1. \end{aligned}$$

Then lifting the variable x_5 yields $\alpha_5 = 1$. This results in a simple LCI: $x_1 + x_2 + x_3 + x_4 + x_5 \leq 2$. If we reverse the ordering, we will obtain a simple LCI: $x_2 + x_3 + x_4 + 2x_5 \leq 2$.

Sequential Up-Lifting

Remark

- As shown by [GNS99], identifying a lifting sequence that leads to the most violated LCI is NP-hard even for simple LCIs.

Proposition (Sequential Down-Lifting, [Gu95])

Suppose (5) is valid for K^1. If $K^0 \neq \emptyset$, then

$$\alpha_1 x_1 + \sum_{i=2}^{n} \alpha_i x_i \leq \beta + \alpha_1 \tag{8}$$

is valid for K, where $\alpha_1 \equiv \xi - \beta$ and $\xi \equiv \max_{x \in K^0} \sum_{i=2}^{n} \alpha_i x_i$. Moreover, if K^1 is full-dimensional and (5) defines a facet of $\text{conv}(K^1)$, then K is full-dimensional and (8) defines a facet of K.

Lifting Sequence

Empirical, what is a good lifting sequence?

- Integer-valued variables should be lifted after fractional variables.
- Determine the sequence based on fractionality or reduced cost.

Variants

- mixed 0-1 knapsack set
- integer knapsack set
- mixed-integer knapsack set [Ata03]
- packing: $\{x \in \{0,1\}^n : a^\top x \geq b\}$

Readers are referred to [HGH$^+$19, Gu95] for more details.

References I

[Ata03] Alper Atamtürk, *On the facets of the mixed–integer knapsack polyhedron*, Mathematical Programming **98** (2003), no. 1, 145–175.

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References II

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